Weekly Assignment 2 (Due Tuesday 7/6 at 11:59PM)

Overview: This assignment is worth **112 points**. Each question is worth **16 points**. Your score for each question will depend on the graders determination of your proficiency in each of the following categories: Conceptual Understanding, Strategies & Reasoning, Computation & Execution, and Communication. You can earn up to 4 points for each category. The grader determines your score for each category using the Weekly Assignment Rubric.

Guidelines: You are required to adhere to the weekly assignment guidelines and the assignment submission guidelines in the syllabus. If you fail to follow the guidelines, you risk receiving no credit for you work. Turn in your assignment via gradescope.

Directions: Complete the following exercises from the Active Calculus textbook. You can click the links below to go directly to the exercise.

1. Exercise 9.4.15. Here's a useful tool for visualizing part (d): GeoGebra: Parallelepiped.

Solution. **a.** To prove this claim, just follow the definitions. We have

$$\begin{aligned}
\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} &= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \\
&= u_1v_2w_3 - u_1w_2v_3 - u_2v_1w_3 + u_2w_1v_3 + u_3v_1w_2 - u_3w_1v_2 \\
&= \langle u_2v_3 - u_3v_2, -(v_1u_3 - v_3u_1), u_1v_2 - u_2v_1 \rangle \cdot \langle w_1, w_2, w_3 \rangle \\
&= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}.
\end{aligned}$$

b. This follows from the anticommutativity of the cross product. We have

$$\begin{aligned}
\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} &= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} && \text{(by (a))} \\
&= -(\mathbf{v} \times \mathbf{u}) \cdot \mathbf{w} && \text{(by anticommutativity)} \\
&= - \begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} && \text{(by (a)).}
\end{aligned}$$

c. This follows from parts (a) and (b). We have

$$\begin{aligned}
(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} && \text{(by (a))} \\
&= \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} && \text{(by b)} \\
&= (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} && \text{(by a).}
\end{aligned}$$

The other claim is similar.

d. **i.** Since $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is the signed volume of the parallelepiped spanned by $\mathbf{u}, \mathbf{v}, \mathbf{w}$, we can conclude that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \neq 0$ if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are not coplanar.

ii. If $\mathbf{u}, \mathbf{w}, \mathbf{w}$ are coplanar, then the parallelpiped has zero volume. Therefore, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = 0$.

iii. It is clear that (i) and (ii) proves both implications.

□

2. Exercise 9.5.11. Here's an updated visualization with the data from the solution: GeoGebra:Exercise 9.5.11.

Solution. **a.** The direction of the line is $\mathbf{u} = \langle -2, 1, 3 \rangle$.

b. Following the definition, the parametric equations are given y $x(t) = -4-2t$, $y(t) = 2+t$ and $z(t) = 17+5t$.

c. The lines are not parallel. Therefore, the lines intersect if and only if there exists a unique $(s, t) \in \mathbb{R}^2$ such that $x(t) = x(s)$, $y(t) = y(s)$, and $z(t) = z(s)$. This is a system of three equations in two variables:

$$\begin{cases} -4-2t = 4-2s \\ 2+t = -2+s \\ 17+5t = 1+3s \end{cases}.$$

By substitution or elimination, you will find that $(s, t) = (2, -2)$ is the unique pair. Therefore, the lines intersect at the points $(0, 0, 7)$.

d. The angle $0 \leq \theta \leq \pi$ between the lines is given by the angle between the direction vectors $\mathbf{u}$ and $\mathbf{v}$. Therefore,

$$\theta = \cos^{-1}\left(\mathbf{u} \cdot \mathbf{v} / |\mathbf{u}||\mathbf{v}|\right) = 12.6\,\mathrm{deg}\,.$$

e. We need to find a point on the plane and a normal vector. A point on the plane is the point of intersection $(0, 0, 7)$ of the two lines since both lines are contained in the plane. A normal vector must be perpendicular to both the lines. Therefore, we can take $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \langle 2, 4, 0 \rangle$ as the normal vector. Then the vector equation of the plane is given by

$$\langle 2, 4, 0 \rangle \cdot \langle x, y, z \rangle = \langle 2, 4, 0 \rangle \cdot \langle 0, 0, 7 \rangle$$

and the scalar equation is given by

$$2x + 4y = 0.$$

□

3. Exercise 9.5.12. It will help to draw a picture.[1]

Solution. **a.** A normal vector is $\mathbf{n} = \langle 4, -5, 1 \rangle$.

b. The plane contains the vectors $\mathbf{u} = \langle 1, 1, 1 \rangle - \langle 0, 1, -1 \rangle = \langle 1, 0, 2 \rangle$ and $\mathbf{v} = \langle 1, 1, 1 \rangle - \langle 4, 2, -1 \rangle = \langle -3, -1, 2 \rangle$. Therefore, $\mathbf{m} = \mathbf{u} \times \mathbf{v} = \langle 2, -8, -1 \rangle$ is a normal vector. Thus, a vector equation of the plane is given by

$$\langle 2, -8, -1 \rangle \cdot \langle x, y, z \rangle = \langle 2, -8, -1 \rangle \cdot \{1, 1, 1\}$$

and a scalar equation is gvein by

$$2x - 8y - z = -7.$$

[1] Once you have drawn the picture for yourself, you can see this visualization of the situation: GeoGebra: Exercise 9.5.12

c. The angle between the planes is the acute angle between the normal vectors. A standard calculation shows that the angle is 29.18 deg.

d. A point (x_0, y_0, z_0) in both planes must satisfy both equations. Therefore, $4x_0 - 5y_0 + 2 + 2x_0 - 8y_0 + 7 = 0$. This simplifies to $6x_0 - 13y_0 = -9$. Pick $(x_0, y_0) = (5, 3)$ which satisfies the equation. Then $z_0 = 2 * 5 - 8 * 3 + 7 = -7$ so that $(5, 3, -7)$ is a point on both planes.

e. A direction vector $\mathbf{w}$ for the line of intersection of the two planes is given by the cross product of their normal vectors. Therefore,

$$\mathbf{w} = \langle 4, -5, 1\rangle \times \langle 2, -8, -1\rangle = \langle 13, 6, -22\rangle.$$

Since $(5, 3, -7)$ is a point in the line, the vector form of the line is given by

$$\mathbf{r}(t) = \langle 5, 3, -7\rangle + t\langle 13, 6, -22\rangle.$$

f. By definition, the parametric equations are given by $x(t) = 5 + 13t$, $y(t) = 3 + 6t$ and $z(t) = -7 - 22t$.

□

4. Let p denote the plane with scalar equation $2x+2y+z = 1$. Let $P = (0, 0, 1)$ and $Q = (2, -1, 1)$. A vector normal to p is $\mathbf{n} = (2, 2, 1)$.

a. Show that P lies in the plane p, but Q does not.

b. Compute $|\text{comp}_{\mathbf{n}}(\overrightarrow{PQ})|$ and explain why this is the shortest distance from Q to the plane p.

Here's the relevant visualization: GeoGebra: Distance from a Point to a Plane.

Solution. **a.** Just check that P satisfies the scalar equation and that Q does not.

b. Note that $\overrightarrow{PQ} = \langle 2, -1, 0\rangle$. Using the formula for the component of this vector along $\mathbf{n}$, we have

$$|\text{comp}_{\mathbf{n}}\overrightarrow{PQ}| = \frac{2}{3}.$$

As we saw in Section 9.3, we can write $\overrightarrow{PQ} = \text{proj}_{\mathbf{n}}\overrightarrow{PQ} + \text{proj}_{\perp\mathbf{n}}\overrightarrow{PQ}$ where $\text{proj}_{\mathbf{n}}\overrightarrow{PQ}$ is a vector parallel to $\mathbf{n}$. Since $\mathbf{n}$ is the normal vector for the plane, this means that $|\text{proj}_{\mathbf{n}}\overrightarrow{PQ}| = |\text{comp}_{\mathbf{n}}\overrightarrow{PQ}|$ is the length of the line segment that is perpendicular to p and joins a (unique) point R in the plane to Q. Geometrically, it is clear that the length of this line segment is the shortest distance from P to p.

□

5. Exercise 9.6.13.

Solution. **a.** A parameterization for the $x = c$ trace of f is given by $\mathbf{r}(t) = \langle c, t, 3c^2 + 4t^2 - 2\rangle$

b. A parameterization for the $y = c$ trace of f is given by $\mathbf{r}(t) = \langle t, c, 3t^2 + 4c^2 - 2\rangle$.

c. A parameterization for the $z = k$ level curve of f is given by $\mathbf{r}(t) = \langle \frac{\sqrt{k+2}}{\sqrt{3}} \cos t, \frac{\sqrt{k+2}}{2} \sin t, k+2\rangle$.

d. here's my wireframe plot of the traces and curves for the values $-2, -1, 0, 1, 2$: GeoGebra: Parabaloid Wireframe. The graph of the function is a parabola. It looks like a bowl that holds an unlimited amount of Wheaties.

□

6. Exercise 9.7.14.

Solution. Here's a visualization of the scenario" GeoGebra: Exercise 9.7.14.

a. Comparing components, we have $1+t = s^2$ and $s(1+t) = 1$ so long as $t \neq -1$. Therefore, $s^3 = 1$ whch implies that $s = 1$. Therefore $t = 0$. A quick check shows that $(s, t) = (1, 0)$ satisfies the third equation. Since the system of equations has a solution, we conclude that the curves intersect. The point of intersection is $r(0) = (1, 1, 1)$.

b. Taking derivatives component wise using ordinary rules from calculus, you will find that

$$\mathbf{r}'(0) = \langle 1, 0, -1 \rangle$$

is the tangent vector to $\mathbf{r}$ at the point of intersection of the curves.

c. Similarly,

$$\mathbf{w}'(1) = \langle 2, 0, 1 \rangle$$

is the tangent vector to the curve $\mathbf{w}$ at the point of intersection.

d. A point on the tangent plane is the point of intersection $(1, 1, 1)$ and the tangent plane contains both tangent lines by definition. Since $\mathbf{r}'(0)$ and $\mathbf{w}'(1)$ are the direction vectors for the tangent lines, a normal vector for the tangent plane is given by $\mathbf{n} = \mathbf{r}'(0) \times \mathbf{w}'(1) = \langle 0, -3, 0 \rangle$. Therefore, a scalar equation for the tangent plane is given by $y = 1$.

□

7. Exercise 9.7.15. For part (d), you are asked to graph something in 3D. Use the GeoGebra 3D Calculator to do it.[2] You do not need to include the graph in your write-up (unless you want to).

Solution. **a.** A parameterization is given by $\mathbf{r}(t) = \langle 3, t, \sqrt{9 + t^2} \rangle$.

b. A parameterization is given by $\mathbf{w}(t) = \langle t, 4, \sqrt{t^2 + 16} \rangle$.

c. As in the preceding problem, it is given by $\mathbf{r}'(4) \times \mathbf{w}'(3) = \langle 0, 1, 4/5 \rangle \times \langle 1, 0, 3/5 \rangle = \langle 3/5, 4/5, -1 \rangle$.

d. A point in the tangent plane is $(3, 4, 5)$. Therefore, the scalar equation of the plane is given by

$$\frac{3}{5}x + \frac{4}{5}y - z = 0.$$

The relevant visualization is here: GeoGebra: Exercise 9.7.15

□

[2] In case you haven't noticed: I think GeoGebra is awesome for visualizing calculus.